《基于有限差分的部分饱和双重孔隙介质弹性波模拟与分析》*的补充材料

石志奇[1)2)3)] 何晓[1)2)3)†] 刘琳[1)3)] 陈德华[1)2)3)]

1)（中国科学院声学研究所，声场声信息国家重点实验室，北京 100190）

2)（中国科学院大学，北京 100049）

3)（北京市海洋深部钻探测量工程技术研究中心，北京 100190）

Santos 两流一固三相介质模型解析解

考虑一个体力源按照组分的体积含量分配在固体相和流体相上时，对文中（3）式和（8）式作傅里叶变换可以得到频域的 Santos 波动方程：

$$\omega^2\left[\tilde{D}_{11}\tilde{\boldsymbol{u}}+\tilde{D}_{12}\tilde{\boldsymbol{U}}^{(1)}+\tilde{D}_{13}\tilde{\boldsymbol{U}}^{(2)}\right]+N\nabla^2\tilde{\boldsymbol{u}}+\nabla\left[\left(A+N\right)\nabla\cdot\tilde{\boldsymbol{u}}+Q_1\nabla\cdot\tilde{\boldsymbol{U}}^{(1)}+Q_2\nabla\cdot\tilde{\boldsymbol{U}}^{(2)}\right]=\left(1-\phi\right)\tilde{\boldsymbol{f}},$$

$$\omega^2\left[\tilde{D}_{12}\tilde{\boldsymbol{u}}+\tilde{D}_{22}\tilde{\boldsymbol{U}}^{(1)}+\tilde{D}_{23}\tilde{\boldsymbol{U}}^{(2)}\right]+\nabla\left[Q_1\nabla\cdot\tilde{\boldsymbol{u}}+R_1\nabla\cdot\tilde{\boldsymbol{U}}^{(1)}+R_3\nabla\cdot\tilde{\boldsymbol{U}}^{(2)}\right]=S_1\phi\tilde{\boldsymbol{f}},$$

$$\omega^2\left[\tilde{D}_{13}\tilde{\boldsymbol{u}}+\tilde{D}_{23}\tilde{\boldsymbol{U}}^{(1)}+\tilde{D}_{33}\tilde{\boldsymbol{U}}^{(2)}\right]+\nabla\left[Q_2\nabla\cdot\tilde{\boldsymbol{u}}+R_3\nabla\cdot\tilde{\boldsymbol{U}}^{(1)}+R_2\nabla\cdot\tilde{\boldsymbol{U}}^{(2)}\right]=S_2\phi\tilde{\boldsymbol{f}},\tag{A1}$$

式中，$\tilde{\boldsymbol{u}}$，$\tilde{\boldsymbol{U}}^{(1)}$，$\tilde{\boldsymbol{U}}^{(2)}$ 和 $\tilde{\boldsymbol{f}}$ 分别表示固体相、非润湿相、润湿相和体力的傅里叶变换．引入势函数 φ_1，φ_2，φ_3，$\boldsymbol{\psi}_1$，$\boldsymbol{\psi}_2$，$\boldsymbol{\psi}_3$，$\varPhi$，$\boldsymbol{\psi}$，则：

$$\begin{aligned}\tilde{\boldsymbol{u}}&=\nabla\varphi_1+\nabla\times\boldsymbol{\psi}_1,\\ \tilde{\boldsymbol{U}}^{(1)}&=\nabla\varphi_2+\nabla\times\boldsymbol{\psi}_2,\\ \tilde{\boldsymbol{U}}^{(2)}&=\nabla\varphi_3+\nabla\times\boldsymbol{\psi}_3,\\ \tilde{\boldsymbol{f}}&=\nabla\Phi+\nabla\times\boldsymbol{\Psi}.\end{aligned}\tag{A2}$$

这里，$\nabla\times\boldsymbol{\psi}_1=\nabla\times\boldsymbol{\psi}_2=\nabla\times\boldsymbol{\psi}_3=\nabla\times\boldsymbol{\psi}=0$．假设声源为胀缩源且 $\varPhi=\delta\left(r\right)s\left(\omega\right)$，将（A2）式代入（A1）式得到非齐次线型方程组：

$$\begin{aligned}
&\omega^2\left[\tilde{D}_{11}\varphi_1+\tilde{D}_{12}\varphi_2+\tilde{D}_{13}\varphi_3\right]+\left[P\nabla^2\varphi_1+Q_1\nabla^2\varphi_2+Q_2\nabla^2\varphi_3\right]=(1-\phi)\delta(r)s(\omega),\\
&\omega^2\left[\tilde{D}_{12}\varphi_1+\tilde{D}_{22}\varphi_2+\tilde{D}_{23}\varphi_3\right]+\left[Q_1\nabla^2\varphi_1+R_1\nabla^2\varphi_2+R_3\nabla^2\varphi_3\right]=S_1\phi\delta(r)s(\omega),\\
&\omega^2\left[\tilde{D}_{13}\varphi_1+\tilde{D}_{23}\varphi_2+\tilde{D}_{33}\varphi_3\right]+\left[Q_2\nabla^2\varphi_1+R_3\nabla^2\varphi_2+R_2\nabla^2\varphi_3\right]=S_2\phi\delta(r)s(\omega),
\end{aligned}\tag{A3}$$

（A3）式可以等效转换为齐次线性方程组与声源处正则化条件描述的问题. 为了得到正则化条件，对（A3）式在一个小的球形区域进行体积分，当球的半径趋于零时，根据高斯定理和球形对称条件，有

$$\begin{aligned}
&\lim_{\sigma\to 0}\int_{s_\sigma}\left(P\frac{\partial\varphi_1}{\partial r}+Q_1\frac{\partial\varphi_2}{\partial r}+Q_2\frac{\partial\varphi_3}{\partial r}\right)\mathrm{d}s=(1-\phi)\delta(r)s(\omega),\\
&\lim_{\sigma\to 0}\int_{s_\sigma}\left(Q_1\frac{\partial\varphi_1}{\partial r}+R_1\frac{\partial\varphi_2}{\partial r}+R_3\frac{\partial\varphi_3}{\partial r}\right)\mathrm{d}s=S_1\phi\delta(r)s(\omega),\\
&\lim_{\sigma\to 0}\int_{s_\sigma}\left(Q_2\frac{\partial\varphi_1}{\partial r}+R_3\frac{\partial\varphi_2}{\partial r}+R_2\frac{\partial\varphi_3}{\partial r}\right)\mathrm{d}s=S_2\phi\delta(r)s(\omega).
\end{aligned}\tag{A4}$$

根据（A3）式和（A4）式可求得：

$$\begin{aligned}
&\varphi_1(r,\ \omega)=\frac{s(\omega)\left(\alpha\mathrm{e}^{-\mathrm{i}k_{\mathrm{p}1}r}+\beta\mathrm{e}^{-\mathrm{i}k_{\mathrm{p}2}r}+\gamma\mathrm{e}^{-\mathrm{i}k_{\mathrm{p}3}r}\right)}{4\pi r},\\
&\varphi_2(r,\ \omega)=\frac{s(\omega)\left(\alpha A_{\mathrm{p}_1}\mathrm{e}^{-\mathrm{i}k_{\mathrm{p}1}r}+\beta A_{\mathrm{p}_2}\mathrm{e}^{-\mathrm{i}k_{\mathrm{p}2}r}+\gamma A_{\mathrm{p}_3}\mathrm{e}^{-\mathrm{i}k_{\mathrm{p}3}r}\right)}{4\pi r},\\
&\varphi_3(r,\ \omega)=\frac{s(\omega)\left(\alpha B_{\mathrm{p}_1}\mathrm{e}^{-\mathrm{i}k_{\mathrm{p}1}r}+\beta B_{\mathrm{p}_2}\mathrm{e}^{-\mathrm{i}k_{\mathrm{p}2}r}+\gamma B_{\mathrm{p}_3}\mathrm{e}^{-\mathrm{i}k_{\mathrm{p}3}r}\right)}{4\pi r},
\end{aligned}\tag{A5}$$

式中，$k_{\mathrm{p}i}$ 为第 i 种纵波的波数. $A_{\mathrm{p}i}$，$B_{\mathrm{p}i}$，α，β，γ 的表达式满足以下关系式：

$$A_{\mathrm{p}i}=-\frac{\begin{vmatrix}V_{\mathrm{p}_i}^2D_{11}-P & V_{\mathrm{p}_i}^2D_{13}-Q_2\\ V_{\mathrm{p}_i}^2D_{12}-Q_1 & V_{\mathrm{p}_i}^2D_{23}-R_3\end{vmatrix}}{\begin{vmatrix}V_{\mathrm{p}_i}^2D_{12}-Q_1 & V_{\mathrm{p}_i}^2D_{13}-Q_2\\ V_{\mathrm{p}_i}^2D_{22}-R_1 & V_{\mathrm{p}_i}^2D_{23}-R_3\end{vmatrix}},\quad B_{\mathrm{p}i}=-\frac{\begin{vmatrix}V_{\mathrm{p}_i}^2D_{11}-P & V_{\mathrm{p}_i}^2D_{12}-Q_1\\ V_{\mathrm{p}_i}^2D_{12}-Q_1 & V_{\mathrm{p}_i}^2D_{22}-R_1\end{vmatrix}}{\begin{vmatrix}V_{\mathrm{p}_i}^2D_{12}-Q_1 & V_{\mathrm{p}_i}^2D_{13}-Q_2\\ V_{\mathrm{p}_i}^2D_{22}-R_1 & V_{\mathrm{p}_i}^2D_{23}-R_3\end{vmatrix}},$$

$$\begin{bmatrix} P+Q_1A_{p_1}+Q_2B_{p_1} & P+Q_1A_{p_2}+Q_2B_{p_2} & P+Q_1A_{p_3}+Q_2B_{p_3} \\ Q_1+R_1A_{p_1}+R_3B_{p_1} & Q_1+R_1A_{p_2}+R_3B_{p_2} & Q_1+R_1A_{p_3}+R_3B_{p_3} \\ Q_2+R_3A_{p_1}+R_2B_{p_1} & Q_2+R_3A_{p_2}+R_2B_{p_2} & Q_2+R_3A_{p_3}+R_2B_{p_3} \end{bmatrix}\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix}=\begin{bmatrix}-s(\omega)(1-\phi)/4\pi\\ -s(\omega)S_1\phi/4\pi\\ -s(\omega)S_2\phi/4\pi\end{bmatrix},$$

当$\eta_{\mathrm{f}}^{(1)}=\eta_{\mathrm{f}}^{(2)}=0$时，对（A5）式进行反傅里叶变换可求得时域点源激发声场的解析式：

$$\begin{aligned}\varphi_1(r,t)&=\frac{\alpha s\left(t-r/V_{p_1}\right)+\beta s\left(t-r/V_{p_2}\right)+\gamma s\left(t-r/V_{p_3}\right)}{4\pi r},\\ \varphi_2(r,t)&=\frac{\alpha A_{p_1}s\left(t-r/V_{p_1}\right)+\beta A_{p_2}s\left(t-r/V_{p_2}\right)+\gamma A_{p_3}s\left(t-r/V_{p_3}\right)}{4\pi r},\\ \varphi_3(r,t)&=\frac{\alpha B_{p_1}s\left(t-r/V_{p_1}\right)+\beta B_{p_2}s\left(t-r/V_{p_2}\right)+\gamma B_{p_3}s\left(t-r/V_{p_3}\right)}{4\pi r}.\end{aligned}\tag{A6}$$

对于沿 y 轴的胀缩线源激发声场解析解，可通过对（A6）式沿 y 轴积分求得

$$\begin{aligned}\varphi_1(x,z,t)&=\frac{\alpha H\left(t-d/V_{p_1}\right)}{2\pi}\int_{d/V_{p_1}}^{t}\frac{s(t-\tau)}{\sqrt{\tau^2-d^2/V_{p_1}^2}}\mathrm{d}\tau\\ &+\frac{\beta H\left(t-d/V_{p_2}\right)}{2\pi}\int_{d/V_{p_2}}^{t}\frac{s(t-\tau)}{\sqrt{\tau^2-d^2/V_{p_2}^2}}\mathrm{d}\tau+\frac{\gamma H\left(t-d/V_{p_3}\right)}{2\pi}\int_{d/V_{p_3}}^{t}\frac{s(t-\tau)}{\sqrt{\tau^2-d^2/V_{p_3}^2}}\mathrm{d}\tau,\\ \varphi_2(x,z,t)&=\frac{\alpha A_{p_1}H\left(t-d/V_{p_1}\right)}{2\pi}\int_{d/V_{p_1}}^{t}\frac{s(t-\tau)}{\sqrt{\tau^2-d^2/V_{p_1}^2}}\mathrm{d}\tau\\ &+\frac{\beta A_{p_2}H\left(t-d/V_{p_2}\right)}{2\pi}\int_{d/V_{p_2}}^{t}\frac{s(t-\tau)}{\sqrt{\tau^2-d^2/V_{p_2}^2}}\mathrm{d}\tau+\frac{\gamma A_{p_3}H\left(t-d/V_{p_3}\right)}{2\pi}\int_{d/V_{p_3}}^{t}\frac{s(t-\tau)}{\sqrt{\tau^2-d^2/V_{p_3}^2}}\mathrm{d}\tau,\\ \varphi_3(x,z,t)&=\frac{\alpha B_{p_1}H\left(t-d/V_{p_1}\right)}{2\pi}\int_{d/V_{p_1}}^{t}\frac{s(t-\tau)}{\sqrt{\tau^2-d^2/V_{p_1}^2}}\mathrm{d}\tau\\ &+\frac{\beta B_{p_2}H\left(t-d/V_{p_2}\right)}{2\pi}\int_{d/V_{p_2}}^{t}\frac{s(t-\tau)}{\sqrt{\tau^2-d^2/V_{p_2}^2}}\mathrm{d}\tau+\frac{\gamma B_{p_3}H\left(t-d/V_{p_3}\right)}{2\pi}\int_{d/V_{p_3}}^{t}\frac{s(t-\tau)}{\sqrt{\tau^2-d^2/V_{p_3}^2}}\mathrm{d}\tau.\end{aligned}$$

(A7)